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Representation and Approximation of Functions via (0, 2)-Interpolation

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DEDICATED TO THE MEMORY OF GÉZA FREUD

1. INTRODUCTION AND STATEMENT OF RESULTS

Consider a doubly indexed sequence of points $x_{n\nu}$ ($n \in \mathbb{N}$, $\nu = 1, 2,..., n$) such that

$$1 \geqslant x_{n1} > x_{n2} > \cdots > x_{nn} \geqslant -1. \tag{1}$$

The problems of *existence*, *uniqueness*, *representation*, *convergence*, etc., of polynomials p_{2n-1} of degree $\leqslant 2n-1$ where the values of p_{2n-1} and those of its second derivative are prescribed at the points (1) were studied by Turán et al. [1–3, 12]. In particular, they found that the zeros

$$1 = \xi_{n1} > \xi_{n2} > \cdots > \xi_{nn} = -1$$

of the polynomial $(1-x^2)\, P'_{n-1}(x)$, where $P_{n-1}(x)$ is the $(n-1)$th Legendre polynomial, are appropriate for this so-called (0, 2)-interpolation problem. In this connection Professor G. Freud [6] proved the following

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THEOREM A. *Let f be a continuous function on $[-1, 1]$ such that*

$$|f(x+h)-2f(x)+f(x-h)| = o(h) \qquad as \quad h \to 0. \tag{2}$$

Denote by $R_n(f; x)$ the $(0, 2)$-interpolation polynomial of Turán et al. satisfying

$$R_n(f; \xi_{n\nu}) = f(\xi_{n\nu}), \qquad R_n''(f; \xi_{n\nu}) = \beta_{n\nu},$$

where

$$|\beta_{n\nu}| \leqslant \varepsilon_n \frac{n}{\sqrt{1-\xi_{n\nu}^2}} \qquad (\nu = 2, 3, ..., n-1),$$

$$|\beta_{n0}| \leqslant \varepsilon_n n^2, \qquad |\beta_{nn}| \leqslant \varepsilon_n n^2,$$

and $\lim_{n\to\infty} \varepsilon_n = 0$. Then $R_n(f; x)$ converges uniformly to $f(x)$ on $[-1, 1]$ as $n \to \infty$.

In several of his papers (see, e.g., [7, 8]) Professor Freud also investigated the problem of approximation on the real line. It is in this spirit that we wish to study the question of (0, 2)-interpolation. As a first result in this direction Kiš [11] proved

THEOREM B. *Let f be a periodic function with period 2π. For every odd n there exists a unique trigonometric polynomial $S_n(f; \gamma, x)$ of the form*

$$a_0 + \sum_{\nu=1}^{n-1} (a_\nu \cos \nu x + b_\nu \sin \nu x) + a_n \cos nx$$

which interpolates f in the points $2\nu\pi/n$ $(\nu = 0, 1, ..., n-1)$ and whose second derivative assumes prescribed values $\gamma_{n\nu}$ at these points. If f satisfies the condition (2) and

$$\gamma_{n\nu} = o(n) \qquad (\nu = 0, 1, ..., n-1),$$

then, as n tends to infinity,

$$S_n(f; \gamma, x) \to f(x)$$

uniformly on the whole real line. The condition (2) cannot be replaced by $f \in \mathrm{Lip}\, \alpha$ with $\alpha \in (0, 1)$, even if the numbers $\gamma_{n\nu}$ are all taken to be zero.

In order to cover the case of non-periodic functions we may use entire functions of exponential type which constitute a natural generalization of trigonometric polynomials (see [4, Theorem 6.10.1]). Introducing the fundamental functions

$$A_n(z) \begin{cases} := \dfrac{\sin \pi z}{\pi z}\left(1 + z\displaystyle\int_0^z \frac{1}{\zeta^2}\left(1 - \frac{\sin \pi\zeta}{\pi\zeta}\right) d\zeta\right) & \text{if } n=0, \\ := (-1)^n \dfrac{\sin \pi z}{\pi(z-n)}\left(\dfrac{z}{n} + (z-n)\displaystyle\int_{-n}^{-n+z} \frac{1}{\zeta^2}\left(1 - \frac{\sin \pi\zeta}{\pi\zeta}\right) d\zeta\right) & \\ \quad - \dfrac{\sin \pi z}{(\pi n)^3}(1 - \cos \pi z) & \text{if } n \neq 0 \end{cases} \tag{3}$$

and

$$B_n(z) \begin{cases} := \dfrac{\sin \pi z}{2\pi}\displaystyle\int_0^z \frac{\sin \pi\zeta}{\pi\zeta}\, d\zeta & \text{if } n=0, \\ := (-1)^n \dfrac{\sin \pi z}{2\pi^2}\displaystyle\int_{-n}^{-n+z}\left(\frac{1}{n} + \frac{1}{\zeta}\right) \sin \pi\zeta\, d\zeta & \text{if } n \neq 0 \end{cases} \tag{4}$$

we define for any $f \in C^2(-\infty, \infty)$ the interpolation operator

$$R(f; z) := \sum_{n=-\infty}^{\infty} (f(n)\, A_n(z) + f''(n)\, B_n(z)) \tag{5}$$

which has the properties (see [10])

(i) $R(f; \cdot)$ is an entire function of exponential type 2π,

(ii) $R(f; n) = f(n)$, $R''(f; n) = f''(n)$ for all integers n,

(iii) $R'(f; 0) = R'''(f; 0) = 0$.

First we consider the problem of representing entire functions of exponential type by this interpolation operator. We obtain

THEOREM 1. *Let f be an entire function of exponential type $\tau < 2\pi$. If for some $\lambda > 1$*

$$\sum_{\nu=-n}^{n} |f(\nu)| = O(n^2(\log n)^{-\lambda})$$

and

$$\sum_{\nu=-n}^{n} |f''(\nu)| = O(n^2(\log n)^{-\lambda}) \tag{6}$$

as $n \to \infty$, then the series (5) *converges absolutely and uniformly on every compact subset of* $\mathbb{C}$ *and*

$$f(z) = R(f; z) + c_{1,\pi}(f) \sin \pi z + c_{2,\pi}(f) \sin 2\pi z, \tag{7}$$

where

$$\begin{aligned} c_{1,\sigma}(f) &:= \frac{1}{3}\left(\frac{4}{\sigma} f'(0) + \frac{1}{\sigma^3} f'''(0)\right) \\ c_{2,\sigma}(f) &:= -\frac{1}{6}\left(\frac{1}{\sigma} f'(0) + \frac{1}{\sigma^3} f'''(0)\right). \end{aligned} \tag{8}$$

Remark 1. The example

$$f(z) = \pi z \sin 2\pi z + \cos 2\pi z - 1 \tag{9}$$

shows that $\tau = 2\pi$ is inadmissible in Theorem 1. Furthermore, condition (6) is best possible in the sense that

$$\sum_{n=-\infty}^{\infty} f''(n)\, B_n(z) \tag{10}$$

does not converge absolutely, if

$$\sum_{\nu=-n}^{n} |f''(\nu)| \geqslant c \cdot \frac{n^2}{\log n} \qquad (c > 0) \tag{11}$$

for a sequence of integers n tending to infinity.

THEOREM 2. *Let f be an entire function of exponential type 2π. If*

$$f(x) = o(x) \qquad \text{as} \quad x \to \pm\infty \tag{12}$$

and if for some $\lambda > 1$ and all integers n

$$|f''(n)| = O(|n| (\log |n|)^{-\lambda}) \qquad \text{as} \quad n \to \pm\infty, \tag{13}$$

then the series (5) *converges absolutely and uniformly on every compact subset of* $\mathbb{C}$ *and* (7) *holds.*

Remark 2. The example in (9) shows that in (12) the o cannot be replaced by O. Furthermore, in (13) the exponent λ cannot in general be allowed to be 1.

As a consequence of Theorem 2 we obtain the

COROLLARY. *Let f be an entire function of exponential type 2π. If for some $\lambda > 1$*

$$|f(x)| = O(|x|(\log |x|)^{-\lambda}) \qquad as \quad x \to \pm\infty, \tag{14}$$

then (7) *holds.*

With the help of Theorem 2 we are able to prove an analogue of Theorem A which may also be looked upon as an extension of Theorem B.

Let f be bounded on the real line and let $(\beta_{\tau n})_{n \in \mathbb{Z}}$ be a bounded sequence of complex numbers depending on a parameter $\tau > 0$. Then (see Lemma 5 below) the series

$$R_\tau(f; \beta, z) := \sum_{n=-\infty}^{\infty} \left(f\left(\frac{n\pi}{\tau}\right) A_n\left(\frac{\tau}{\pi} z\right) + \left(\frac{\pi}{\tau}\right)^2 \beta_{\tau n} B_n\left(\frac{\tau}{\pi} z\right)\right) \tag{15}$$

converges absolutely and uniformly on every compact subset of $\mathbb{C}$ and represents an entire function of exponential type 2τ such that

$$R_\tau\left(f; \beta, \frac{n\pi}{\tau}\right) = f\left(\frac{n\pi}{\tau}\right), \qquad R_\tau''\left(f; \beta, \frac{n\pi}{\tau}\right) = \beta_{\tau n} \qquad (n = 0, \pm 1, \pm 2, ...).$$

We now have

THEOREM 3. *If $f: \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function satisfying* (2) *uniformly in x on the real line and*

$$\sup_n |\beta_{\tau n}| = o(\tau) \qquad as \quad \tau \to \infty, \tag{16}$$

then

$$\lim_{\tau \to \infty} R_\tau(f; \beta, x) = f(x)$$

uniformly in x on every compact subset of the real line. The condition (2) *cannot be replaced by*

$$f \in \operatorname{Lip} \alpha \tag{17}$$

with $\alpha \in (0, 1)$, even if the numbers $\beta_{\tau n}$ are all taken to be zero.

Remark 3. Previously (see [10]) we had obtained the conclusion of Theorem 3 under the additional condition that

$$(1 + |x|^\lambda)\, |f(x)| \leqslant 1 \tag{18}$$

for some $\lambda > 0$ and all real x.

In order to see the relationship between Theorem 3 and Theorem B let f be of period 2π. For odd n consider the trigonometric polynomial of Kiš satisfying

$$S_n\left(f; \gamma, \frac{2\nu\pi}{n}\right) = f\left(\frac{2\nu\pi}{n}\right), \qquad S_n''\left(f; \gamma, \frac{2\nu\pi}{n}\right) = \gamma_{n\nu} \qquad (\nu = 0, 1, ..., n-1).$$

Now set

$$\beta_{n/2, \nu + jn} := \gamma_{n\nu} \qquad (\nu = 0, 1, ..., n-1; j = 0, \pm 1, \pm 2, ...).$$

Applying Theorem 2 to $S_n(f; \gamma, \cdot)$ we see that

$$S_n(f; \gamma, x) \equiv R_{n/2}(f; \beta, x) + c_{1n} \sin \frac{n}{2} x + c_{2n} \sin nx, \tag{19}$$

where with the notation in (8)

$$c_{jn} := c_{j,n/2}(S_n(f; \gamma, \cdot)) \qquad (j = 1, 2).$$

Under the assumptions of Kiš, namely, (2) and

$$\max_\nu |\gamma_{n\nu}| = o(n) \qquad \text{as} \quad n \to \infty,$$

it can be shown that

$$S_n'(f; \gamma, 0) = o(n), \qquad S_n'''(f; \gamma, 0) = o(n^3),$$

which implies $c_{jn} \to 0$ as $n \to \infty$. Now (19) in conjunction with Theorem 3 yields $S_n(f; \gamma, x) \to f(x)$, uniformly on $[0, 2\pi]$ and hence, due to the periodicity, on the whole real line. That Theorem 3 does not constitute a direct generalization of Theorem B is attributable to our normalization (iii) of the function $R_{n/2}(f; \beta, \cdot)$.

2. Lemmas

Lemma 1. *Let G be holomorphic and of exponential type τ in the closed upper half plane. If for some real numbers λ and μ*

$$|G(x)| = O(|x|^\mu (\log |x|)^\lambda) \qquad as \quad x \to \pm\infty, \tag{20}$$

then

$$|G(re^{i\theta})| = O(r^\mu (\log r)^\lambda e^{\tau r \sin\theta}) \qquad as \quad r \to \infty$$

uniformly for $\theta \in [0, \pi]$.

Proof. Apply [4, Theorem 6.2.4] to the function

$$H: z \mapsto (z+i)^{-\mu}(\log(z+2i))^{-\lambda}\, G(z)$$

which is of exponential type τ in the closed upper half plane and bounded on the whole real line.

LEMMA 2. *If G is an entire function of exponential type such that for some real numbers λ and μ*

$$|G(x)| = O(|x|^{\mu}(\log|x|)^{\lambda}) \qquad as \quad x \to \pm\infty,$$

then also

$$|G'(x)| = O(|x|^{\mu}(\log|x|)^{\lambda}) \qquad as \quad x \to \pm\infty.$$

Proof. According to Cauchy's integral formula for the derivative

$$G'(x) = \frac{1}{2\pi}\int_0^{2\pi} G(x+e^{i\phi})\, e^{-i\phi}\, d\phi$$

and the desired result becomes an obvious consequence of the preceding lemma.

LEMMA 3. *Let F be an entire function of exponential type less than 2π. If G is an entire function of exponential type 2π satisfying (20) with $\mu > 0$, $\lambda < 0$ for which*

$$G(n) = F(n), \qquad G''(n) = F''(n) \qquad (n = 0, \pm 1, \pm 2, ...),$$

then

$$F(z) - G(z) = \left(a + \int_0^z \psi(t)\sin \pi t\, dt\right)\sin \pi z, \tag{21}$$

where a is a constant and ψ is a polynomial of degree less than μ.

Proof. Put

$$\kappa(z) := F(z) - G(z). \tag{22}$$

Then κ is an entire function of exponential type such that

$$\kappa(v) = \kappa''(v) = 0 \qquad (v = 0, \pm 1, \pm 2, ...).$$

This implies (see [9, Lemma 1]) that

$$\kappa(z) = \phi(z)\sin \pi z \tag{23}$$

and in turn

$$\phi'(z) = \psi(z) \sin \pi z, \tag{24}$$

where ϕ and ψ are entire functions of exponential type. Thus we obtain the representation

$$\kappa(z) = \left(\phi(0) + \int_0^z \psi(t) \sin \pi t \, dt \right) \sin \pi z.$$

Using (22)–(24) we may also write ψ in the form

$$\psi(z) = \psi_1(z) - \psi_2(z)$$

where

$$\psi_1(z) = \frac{F'(z) - \pi \cos \pi z F(z)/\sin \pi z}{\sin^2 \pi z}$$

and

$$\psi_2(z) = \frac{G'(z) - \pi \cos \pi z G(z)/\sin \pi z}{\sin^2 \pi z}.$$

For $\gamma \in \{\pm\pi/4, \pm 3\pi/4\}$ we readily see that

$$\lim_{r \to \infty} \psi_1(re^{i\gamma}) = 0.$$

Using Lemmas 1 and 2 we also obtain

$$|\psi_2(re^{i\gamma})| = O(r^\mu (\log r)^\lambda) \qquad \text{as} \quad r \to \infty.$$

Now let k be the largest integer smaller than μ and consider

$$\tilde{\psi}(z) := \frac{1}{z^{k+1}} \left(\psi(z) - \sum_{j=0}^{k} \frac{\psi^{(j)}(0)}{j!} z^j \right).$$

An obvious application of [4, Theorem 1.4.2] shows that $\tilde{\psi}$ is bounded in the whole plane. By Liouville's theorem it is therefore a constant, which must be zero since

$$\lim_{r \to \infty} \tilde{\psi}(re^{i\gamma}) = 0 \qquad \text{for} \quad \gamma \in \{\pm\pi/4, \pm 3\pi/4\}.$$

This shows that $\psi(z)$ is a polynomial of degree k.

Notation. For the remainder of this paper $c_1, c_2, \ldots$ will always denote appropriate positive constants.

LEMMA 4. *For $x \in \mathbb{R}$ let n_x be the larger of the possibly two integers closest to x and denote by $N(x)$ the set of all integers between 0 and n_x (including both 0 and n_x). Then for $z = x + iy$ ($x, y \in \mathbb{R}$) the fundamental functions* (3) *and* (4) *may be estimated as follows:*

$$|A_n(z)| \leqslant c_1 e^{2\pi|y|}, \qquad \text{if} \quad n \in \{0, n_x\}, \tag{25}$$

$$|A_n(z)| \leqslant c_2 e^{\pi|y|} + c_3 \left(\frac{1}{|n|^3} + \frac{1}{|n - n_x|^3} \right) (e^{2\pi|y|} - 1),$$
$$\text{if} \quad n \in N(x) \setminus \{0, n_x\}, \tag{26}$$

$$|A_n(z)| \leqslant c_4 \left| \frac{1}{n^3} - \frac{1}{(n-x)^3} \right| e^{\pi|y|}$$
$$+ c_5 \max_{0 \leqslant t \leqslant y} \left| \frac{1}{n^3} - \frac{1}{(n - x + it)^3} \right| (e^{2\pi|y|} - 1),$$
$$\text{if} \quad n \notin N(x); \tag{27}$$

$$|B_n(z)| \leqslant c_1 e^{2\pi|y|}, \qquad \text{if} \quad n \in \{0, n_x\}, \tag{28}$$

$$|B_n(z)| \leqslant c_6 e^{\pi|y|} + c_7 \frac{|z|}{|n(n - n_x)|} (e^{2\pi|y|} - 1),$$
$$\text{if} \quad n \in N(x) \setminus \{0, n_x\}, \tag{29}$$

$$|B_n(z)| \leqslant c_7 \frac{|z|}{|n(n - n_x)|} e^{2\pi|y|}, \qquad \text{if} \quad n \notin N(x). \tag{30}$$

Proof. Since the fundamental functions A_n and B_n are of exponential type 2π and bounded on the real line, independently of n, it is clear that estimates of the form (25) and (28) hold.

Next, we split the integrals in (3) and (4) as

$$\int_{-n}^{-n+z} \cdots = \int_{-n}^{-n+x} \cdots + \int_{-n+x}^{-n+x+iy} \cdots. \tag{31}$$

The first integral on the right-hand side remains bounded for all n and x. As regards the second integral, it can be estimated by constant multiples of

$$1 + \frac{1}{|n - n_x|^3} (e^{\pi|y|} - 1) \qquad \text{and} \qquad \frac{|z|}{|n(n - n_x)|} (e^{\pi|y|} - 1)$$

for $A_n(z)$ and $B_n(z)$, respectively, provided n is different from 0 and n_x. Now (26) and (29) are readily obtained.

Finally, for $n \notin N(x)$ it is quickly seen that in the case of $B_n(z)$ the first integral on the right-hand side of (31) is bounded by a constant multiple of

$|z|/|n(n-n_x)|$. Together with our estimate for the second integral we obtain (30).

For $n \notin N(x)$ the point zero can never lie in the range of integration in (31). Therefore we may write

$$A_n(z) = \frac{(-1)^{n+1} \sin \pi z}{\pi^2} \int_{-n}^{n+z} \frac{\sin \pi\zeta}{\zeta^3}\, d\zeta - \frac{\sin \pi z}{\pi^2 n^3} \int_0^z \sin \pi\zeta\, d\zeta$$

$$= -\frac{\sin \pi z}{\pi^2} \int_0^z \left(\frac{1}{n^3} - \frac{1}{(n-\zeta)^3}\right) \sin \pi\zeta\, d\zeta. \tag{32}$$

Splitting the integral as

$$\int_0^z \cdots = \int_0^x \cdots + \int_x^{x+iy} \cdots$$

we use the second law of the mean for the first integral on the right-hand side to obtain

$$\left| \int_0^x \left(\frac{1}{n^3} - \frac{1}{(n-\zeta)^3}\right) \sin \pi\zeta\, d\zeta \right|$$

$$= \left| \frac{1}{n^3} - \frac{1}{(n-x)^3} \right| \cdot \left| \int_\eta^x \sin \pi\zeta\, d\zeta \right|$$

$$\leqslant \frac{2}{\pi} \left| \frac{1}{n^3} - \frac{1}{(n-x)^3} \right| \qquad \text{where} \quad \eta \in (0, x).$$

This leads to the first term on the right-hand side of (27). The second one is obtained in an obvious way by estimating the integral from x to $x+iy$.

LEMMA 5. *Let $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$ be two sequences of complex numbers. Suppose that for some $\lambda > 1$*

$$\sum_{\nu=-n}^{n} |a_\nu| = O(n^p (\log n)^{-\lambda}), \qquad 0 < p \leqslant 4 \tag{33}$$

and

$$\sum_{\nu=-n}^{n} |b_\nu| = O(n^q (\log n)^{-\lambda}), \qquad 0 < q \leqslant 2 \tag{34}$$

as $n \to \infty$. Then

$$H(z) := \sum_{n=-\infty}^{\infty} (a_n A_n(z) + b_n B_n(z)) \tag{35}$$

converges absolutely and uniformly on every compact subset of $\mathbb{C}$ *and represents an entire function of exponential type* 2π. *Furthermore,*

$$|H(x)| = O(|x|^{\sigma}(\log |x|)^{-\lambda}) \qquad as \quad x \to \pm\infty \tag{36}$$

where $\sigma = \max\{p, q\}$.

Proof. Let C be any compact subset of $\mathbb{C}$ so that there exists an integer k with

$$C \subset \{z \in \mathbb{C} \colon |z| \leqslant k\}.$$

In view of Lemma 4 we have for all $z \in C$

$$|A_n(z)| \leqslant c_8, \qquad |B_n(z)| \leqslant c_8,$$

if $|n| < 2k$, and

$$|A_n(z)| \leqslant c_9 \frac{1}{n^4}, \qquad |B_n(z)| \leqslant c_9 \frac{1}{n^2},$$

if $|n| \geqslant 2k$. To prove the absolute and uniform convergence of the series (35) on C it is now sufficient to show that

$$\sum_{|n| \geqslant 2k} \frac{|a_n|}{n^4} \qquad \text{and} \qquad \sum_{|n| \geqslant 2k} \frac{|b_n|}{n^2}$$

converge; but this can be readily done via Abel's summation. Hence H represents an entire function which must be of exponential type 2π as is seen from Lemma 4.

Let us now verify (36). Without loss of generality we may assume that $x > 0$. Then

$$\sum_{n=0}^{n_x} |a_n A_n(x) + b_n B_n(x)| = O(n_x^{\sigma}(\log n_x)^{-\lambda})$$

as $x \to \infty$. Using Lemma 4 it remains to estimate

$$S_1 := \sum_{n \notin N(x)} \left| \frac{1}{n^3} - \frac{1}{(n-x)^3} \right| \cdot |a_n|$$

and

$$S_2 := \sum_{n \notin N(x)} \frac{x}{|n(n-n_x)|} \cdot |b_n|.$$

For this we split the two summations as

$$\sum_{n=-2n_x+1}^{-1} \cdots + \sum_{n=n_x+1}^{2n_x-1} \cdots + \sum_{n=-\infty}^{-2n_x} \cdots + \sum_{n=2n_x}^{\infty} \cdots.$$

In the case of S_1 the first two sums are obviously of order $O(n_x^p(\log n_x)^{-\lambda})$. For all the indices n in the remaining two

$$\left|\frac{1}{n^3}-\frac{1}{(n-x)^3}\right| \leqslant c_{11}\frac{x}{n^4} \leqslant c_{11}\frac{x^{p-3}}{n^p}$$

and so Abel's summation shows that these sums are of order $o(x^{p-3})$. In the case of S_2 it is sufficient to estimate only the contribution coming from positive indices. For this we write

$$\sum_{n=n_x+1}^{2n_x-1} \frac{x}{n(n-n_x)}\cdot|b_n| \leqslant \sum_{n=1}^{n_x-1} \frac{1}{n}\,|b_{n+n_x}| = O(n_x^q(\log n_x)^{-\lambda})$$

and

$$\sum_{n=2n_x}^{\infty} \frac{x}{n(n-n_x)}\cdot|b_n| \leqslant \sum_{n=n_x}^{\infty} \frac{x^{q-1}}{n^q}\cdot|b_{n+n_x}| = o(x^{q-1}),$$

where in the second case Abel's summation is used in the last step. This completes the proof of (36).

LEMMA 6. *Let $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$ be two sequences of complex numbers. Suppose that*

$$a_n = o(n) \tag{37}$$

and for some $\lambda > 1$

$$|b_n| = O(|n|(\log|n|)^{-\lambda}) \tag{38}$$

as $n \to \pm\infty$. Then the series $H(z)$ defined in (35) represents an entire function of exponential type 2π. Furthermore, for $\theta \in (0, \pi)$

$$\left.\begin{matrix} H(re^{\pm i\theta}) \\ H'(re^{\pm i\theta}) \end{matrix}\right\} = o(re^{2\pi r \sin\theta}) \tag{39}$$

as $r \to \infty$.

Proof. Since (37) and (38) imply (33) and (34), respectively (with $q = 2$ and an arbitrary $p > 2$), we know from Lemma 5 that H represents an entire function of exponential type 2π. It remains to verify (39).

Applying [4, Theorem 1.4.2] to the function

$$z \mapsto \frac{f(z)\, e^{2\pi i z}}{z+iK}, \qquad f \in \{H, H'\}$$

with an appropriate $K>0$ we see that it is enough to prove (39) for $\theta \neq \pi/2$. Then, for $x+iy=re^{\pm i\theta}$ we always have $x \to \pm\infty$ with $r \to \infty$.

Let us first turn to $H(re^{\pm i\theta})$. Referring to Lemma 4 we need only consider those terms in the bounds for the fundamental functions which carry a factor $e^{2\pi|y|}$ since they dominate all other terms as $r \to \infty$. Then in view of (25)–(30) it suffices to show that for $n_x > 0$

$$\sum_{\substack{n=-\infty \\ n \notin \{0, n_x\}}}^{\infty} \left(\frac{1}{|n|^3} + \frac{1}{|n-n_x|^3} \right) |a_n| = o(x) \tag{40}$$

and

$$\sum_{\substack{n=-\infty \\ n \notin \{0, n_x\}}}^{\infty} \frac{1}{|n(n-n_x)|} \cdot |b_n| = o(1) \tag{41}$$

as $x \to \infty$. In the case of (40) we split the summation as

$$\sum_{\substack{n=1 \\ n \neq n_x}}^{2n_x - 1} \cdots + \sum_{n=2n_x}^{\infty} \cdots + \sum_{n=-\infty}^{-1} \cdots.$$

Now using (37) we readily see that the first sum is of order $o(n_x)$, whereas the last two series are even bounded. Hence (40) holds.

In the case of (41) the biggest possible contribution can come from positive indices only, if $n_x > 0$. Let us therefore consider

$$S := \sum_{\substack{n=2 \\ n \neq n_x}}^{\infty} \frac{1}{|n(n-n_x)|} \cdot |b_n|$$

and use the estimate

$$|b_n| \leq c_{12} n (\log n)^{-\lambda}$$

for $n \geq 2$ to obtain

$$S \leq c_{12} \left(\sum_{\substack{n=2 \\ n \neq n_x}}^{2n_x} \frac{1}{|n(n-n_x)|} \cdot 2n_x (\log(2n_x))^{-\lambda} + \sum_{n=2n_x+1}^{\infty} \frac{1}{n-n_x} (\log n)^{-\lambda} \right).$$

It can be shown (see [10, Lemma 3]) that

$$\sum_{\substack{n=-\infty \\ n \notin \{0, n_x\}}}^{\infty} \frac{1}{|n(n-n_x)|} = O\left(\frac{\log n_x}{n_x}\right).$$

Hence the first sum is of order $O((\log n_x)^{1-\lambda})$ and so in particular $o(1)$. For the second sum we have

$$\sum_{n=2n_x+1}^{\infty} \frac{1}{n-n_x} (\log n)^{-\lambda} \leqslant \sum_{n=n_x+1}^{\infty} \frac{1}{n} (\log n)^{-\lambda} = o(1)$$

as $x \to \infty$. Thus (41) is also verified.

To estimate $H'(re^{\pm i\theta})$ we first differentiate the fundamental functions. Considering then only those contributions which grow at least as fast as $e^{2\pi|y|}$we see again that the desired result follows from (40) and (41).

LEMMA 7. *Let χ be a function defined on $\mathbb{R}$ and suppose that it is continuous and bounded. For $\delta > 0$ let*

$$g_\delta(z) := \left(\frac{\sin \delta z/4}{z}\right)^4 \tag{42}$$

and define

$$h(z) := \omega^{-1} \int_{-\infty}^{\infty} g_\delta(t-z)\, \chi(t)\, dt \tag{43}$$

where

$$\omega := \int_{-\infty}^{\infty} g_\delta(t)\, dt.$$

Then h is an entire function of exponential type δ. Furthermore,

$$|h(x)| = O(x^{-2}) \qquad as \quad x \to \pm\infty, \tag{44}$$

if

$$|\chi(x)| = O(x^{-2}) \qquad as \quad x \to \pm\infty. \tag{45}$$

Proof. We only verify the growth property (44) since everything else is well known [13, pp. 257–259].

For $z = x \in \mathbb{R}$ we may write

$$h(x) = \omega^{-1} \int_{-\infty}^{\infty} g_\delta(t)\, \chi(t+x)\, dt.$$

Expressing x^2 as

$$x^2 = (t+x)^2 + t^2 - 2(x+t)\,t$$

we obtain

$$x^2 h(x) = \omega^{-1} \int_{-\infty}^{\infty} g_\delta(t)[(t+x)^2 \chi(t+x)]\,dt + \omega^{-1} \int_{-\infty}^{\infty} t^2 g_\delta(t)\,\chi(t+x)\,dt$$

$$-2\omega^{-1} \int_{-\infty}^{\infty} t g_\delta(t)[(t+x)\,\chi(t+x)]\,dt.$$

Now we see that under condition (45) all three terms on the right-hand side are bounded on the real line.

LEMMA 8. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of complex numbers. If for some $\lambda > 1$*

$$\sum_{\nu=1}^{n} |a_\nu| = O(n^2(\log n)^{-\lambda})$$

and

$$|b_n| = O(n^{-2})$$

as $n \to \infty$, then

$$\sum_{\nu=1}^{n} |a_\nu b_\nu| = O(1).$$

Proof. In view of the estimate $|b_n| \leqslant c_{13} n^{-2}$ it is enough to consider

$$\sum_{\nu=1}^{n} |a_\nu|\,\nu^{-2}.$$

Now the desired result follows via Abel's summation and the fact that

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^\lambda}$$

converges for $\lambda > 1$.

3. PROOFS OF THE RESULTS

Proof of Theorem 1. It follows from Lemma 5 that $R(f;\cdot)$ represents an entire function of exponential type 2π such that

$$|R(f;x)| = O(x^2(\log|x|)^{-\lambda}) \qquad \text{as} \quad x \to \pm\infty.$$

Now, setting $F := f$ and $G := R(f; \cdot)$ we conclude from Lemma 3 that ψ is a polynomial of degree at most 1. In view of (21) this implies that

$$|F(x) - G(x)| = O(|x|) \qquad \text{as} \quad x \to \pm\infty.$$

Hence

$$|f(x)| = O(x^2(\log|x|)^{-\lambda}) \qquad \text{as} \quad x \to \pm\infty$$

and by Lemma 2 also

$$|f'(x)| = O(x^2(\log|x|)^{-\lambda}) \qquad \text{as} \quad x \to \pm\infty. \tag{46}$$

Next we choose $\delta \in (0, (2\pi - \tau)/2)$ and define

$$\chi_m(x) := \frac{1}{1 + (x/m)^2}.$$

Now, if

$$h_m(z) := \omega^{-1} \int_{-\infty}^{\infty} g_\delta(t - z)\, \chi_m(t)\, dt,$$

then, according to Lemma 7,

$$f_m \colon z \mapsto f(z)\, h_m(z)$$

as well is of exponential type less than 2π. Lemmas 7 and 8 show that

$$\sum_{\nu=-n}^{n} |f_m(\nu)| = O(1) \qquad \text{as} \quad n \to \infty. \tag{47}$$

In order to calculate the second derivatives of f_m at the integers we may write for real x

$$h_m^{(j)}(x) = \omega^{-1} \int_{-\infty}^{\infty} g_\delta(t)\, \chi_m^{(j)}(t + x)\, dt \qquad (j = 0, 1, 2) \tag{48}$$

and deduce that for $m \to \infty$

$$\begin{aligned} &|h_m(x)| = O(1), && |x h_m'(x)| = O(1), \\ &|h_m'(x)| = O\left(\frac{1}{m}\right), && |h_m''(x)| = O\left(\frac{1}{m^2}\right) \end{aligned} \tag{49}$$

uniformly in x. Furthermore, transforming (48) into

$$h_m^{(j)}(x) = \omega^{-1} \int_{-\infty}^{\infty} g_\delta(t - x)\, \chi_m^{(j)}(t)\, dt \qquad (j = 0, 1, 2)$$

we conclude with the help of Lemma 7 that for every fixed m

$$|h_m^{(j)}(x)| = O(x^{-2}) \qquad (j=0, 1, 2) \tag{50}$$

as $x \to \pm\infty$. Since

$$f_m''(x) = f''(x)\, h_m(x) + 2f'(x)\, h_m'(x) + f(x)\, h_m''(x)$$

we may use (46), (50), and Lemma 8 to see that for every fixed m

$$\sum_{v=-n}^{n} |f_m''(v)| = O\left(\sum_{v=3}^{n} (\log v)^{-\lambda}\right) = O\left(\frac{n}{(\log n)^{\lambda}}\right) \tag{51}$$

as $n \to \infty$. Now Lemma 5 yields

$$|R(f_m; x)| = O(|x|(\log |x|)^{-\lambda}) \qquad \text{as} \quad x \to \pm\infty.$$

Hence for $F := f_m$ and $G := R(f_m; \cdot)$ in Lemma 3 the corresponding function ψ is a constant. This gives (by Property (iii) of $R(f_m; \cdot)$)

$$f_m(z) = R(f_m, z) + c_{1,\pi}(f_m) \sin \pi z + c_{2,\pi}(f_m) \sin 2\pi z.$$

Obviously,

$$\lim_{m \to \infty} f_m(z) = f(z)$$

uniformly on every compact subset of $\mathbb{C}$ and

$$\lim_{m \to \infty} c_{j,\pi}(f_m) = c_{j,\pi}(f) \qquad \text{for} \quad j = 1, 2.$$

Hence it remains to show that

$$\lim_{m \to \infty} R(f_m; z) = R(f, z).$$

Let C be any compact subset of $\mathbb{C}$ and let ε be a given positive number. By virtue of Lemma 5 and (49) we can find an $n_0 > 0$ such that

$$S_1 := \left|\left(\sum_{v=-\infty}^{-n_0} + \sum_{v=n_0}^{\infty}\right) (f(v)\, A_v(z) + f''(v)\, B_v(z))\right| < \frac{\varepsilon}{6}$$

and

$$S_2 := \left|\left(\sum_{v=-\infty}^{-n_0} + \sum_{v=n_0}^{\infty}\right) (f_m(v)\, A_v(z) + f_m''(v)\, B_v(z))\right| < \frac{\varepsilon}{6}$$

for all $z \in C$ and all positive integers m. Furthermore, there exists a constant $K > 0$ such that for all $z \in C$

$$\sum_{\nu=-\infty}^{\infty} |f(\nu) A_\nu(z)| < K \qquad \text{and} \qquad \sum_{\nu=-\infty}^{\infty} (|f''(\nu)| + 1) |B_\nu(z)| < K.$$

Since $\lim_{m \to \infty} h_m(x) \equiv 1$ we can achieve that for sufficiently large m, say, $m \geqslant m_0$, and $\nu = -n_0 + 1, -n_0 + 2, ..., n_0 - 2, n_0 - 1$

$$|1 - h_m(\nu)| < \frac{\varepsilon}{6K}, \qquad |2f'(\nu) h'_m(\nu)| < \frac{\varepsilon}{6K},$$

and

$$|f(\nu) h''_m(\nu)| < \frac{\varepsilon}{6K}.$$

Hence for all $z \in C$ and all $m \geqslant m_0$

$$\begin{aligned} |R(f; z) - R(f_m; z)| \leqslant \sum_{\nu=-n_0+1}^{n_0-1} & (|f(\nu)(1 - h_m(\nu)) A_\nu(z)| \\ & + |f''(\nu)(1 - h_m(\nu)) B_\nu(z)| + |2f'(\nu) h'_m(\nu) B_\nu(z)| \\ & + |f(\nu) h''_m(\nu) B_\nu(z)|) + S_1 + S_2 < \varepsilon. \end{aligned}$$

Thus the desired representation is proved.

Let us now justify Remark 1. Note that for $n \neq 0$

$$-B_n\left(\frac{1}{2}\right) = \frac{-1}{2\pi} \int_0^{1/2} \left(\frac{1}{n} + \frac{1}{t-n}\right) \sin \pi t \, dt > c_{14} n^{-2}. \tag{52}$$

Hence

$$\sum_{\substack{\nu=-n \\ \nu \neq 0}}^{n} \left| f''(\nu) B_\nu\left(\frac{1}{2}\right)\right| > c_{14} \sum_{\substack{\nu=-n \\ \nu \neq 0}}^{n} |f''(\nu)| \cdot \nu^{-2};$$

but Abel's summation shows that the right-hand side tends to infinity with n, if (11) holds.

Proof of Theorem 2. With the help of [4, Theorem 1.4.4] it is readily verified (see the proofs of Lemmas 1 and 2) that for all $\theta \in (0, \pi)$ and $j = 0, 1$

$$|f^{(j)}(re^{\pm i\theta})| = o(re^{2\pi r \sin \theta}) \qquad \text{as} \quad r \to \infty.$$

By Lemma 6, $R(f;z)$ exists and satisfies together with its first derivative the same growth condition. Setting $F:=0$ and $G:=R(f;\cdot)-f$ the corresponding function ψ in (21) becomes

$$\psi(z)=\frac{\pi\cos\pi z G(z)/\sin\pi z - G'(z)}{\sin^2\pi z}.$$

From this it is seen that for an arbitrary $\gamma\in(0,\pi/2)$

$$|\psi(re^{i\theta})|=o(r)\qquad \text{as}\quad r\to\infty \tag{53}$$

for $\theta=-\gamma,\gamma,\pi-\gamma,\pi+\gamma$. By the Phragmén–Lindelöf principle (53) holds uniformly for all $\theta\in[0,2\pi]$. According to a refined version of Liouville's theorem ψ must be a constant. Now the desired representation follows from (21) by carrying out the integration and taking into account that $R'(f;0)=R'''(f;0)=0$.

To justify the unexplained part of Remark 2 it is enough to show that

$$\sum_{\nu=-\infty}^{\infty}\left|f''(\nu)\,B_\nu\left(\frac{1}{2}\right)\right|=\infty,$$

if

$$|f''(n)|\geq c_{15}\cdot\frac{|n|}{\log|n|},$$

a fact easily seen with the help of (52).

Proof of the Corollary. Obviously (14) implies (12); Lemma 2 shows that it also implies (13).

Proof of Theorem 3. In [10, Lemma 8] we constructed a sequence of entire functions $T_\tau(f;\cdot)$ of exponential type 2τ such that

$$f(x)-T_\tau(f;x)=o(1/\tau),\qquad T'_\tau(f;x)=o(\log\tau),$$
$$T''_\tau(f;x)=o(\tau),\qquad T'''_\tau(f;x)=o(\tau^2)$$

uniformly in x as $\tau\to\infty$. Now denote by $R_\tau(f;\cdot)$ the operator in (5) transformed from integer nodes to $n\pi/\tau$ $(n\in\mathbb{Z})$, i.e.,

$$R_\tau(f;x):=\sum_{n=-\infty}^{\infty}\left(f\left(\frac{n\pi}{\tau}\right)A_n\left(\frac{\tau}{\pi}x\right)+\left(\frac{\pi}{\tau}\right)^2 f''\left(\frac{n\pi}{\tau}\right)B_n\left(\frac{\tau}{\pi}x\right)\right).$$

The crucial observation is that according to Theorem 2

$$T_\tau(f;x)\equiv R_\tau(T_\tau(f;\cdot);x)+c_{1,\tau}(T_\tau(f;\cdot))\sin\tau x+c_{2,\tau}(T_\tau(f;\cdot))\sin 2\tau x$$

holds, if f is simply assumed to be bounded (so that (18) is no longer needed). Using the decomposition

$$\begin{aligned} f(x)-R_\tau(f;\beta,x) &= f(x)-T_\tau(f;x)+T_\tau(f;x)-R_\tau(f;\beta,x) \\ &= f(x)-T_\tau(f;x) \\ &\quad +\sum_{n=-\infty}^{\infty}\left(T_\tau\left(f;\frac{n\pi}{\tau}\right)-f\left(\frac{n\pi}{\tau}\right)\right)A_n\left(\frac{\tau}{\pi}x\right) \\ &\quad +\left(\frac{\pi}{\tau}\right)^2\sum_{n=-\infty}^{\infty}\left(T''_\tau\left(f;\frac{n\pi}{\tau}\right)-\beta_{\tau n}\right)B_n\left(\frac{\tau}{\pi}x\right) \\ &\quad +c_{1,\tau}(T_\tau(f;\cdot))\sin\tau x+c_{2,\tau}(T_\tau(f;\cdot))\sin 2\tau x \qquad (54) \end{aligned}$$

the proof of the convergence is completed as in [10, p. 199].

It remains to show that (2) cannot be replaced by (17). This may be done as follows. For $n\neq 0$ we can write

$$(-1)^n A_n(x)=\frac{\sin\pi x}{\pi}\int_{-n}^{-n+x}\frac{1}{\zeta^2}\left(1-\frac{\sin\pi\zeta}{\pi\zeta}\right)d\zeta+K_n(x),$$

where

$$K_n(x)=\frac{x}{n\pi}\frac{\sin\pi x}{x-n}+(-1)^{n+1}\frac{\sin\pi x}{(\pi n)^3}(1-\cos\pi x).$$

It is easily seen that

$$\sum_{n=-\infty}^{\infty}|K_n(x)|=o(x) \qquad \text{as} \quad x\to\pm\infty.$$

Furthermore,

$$\operatorname{sgn}\int_{-n}^{-n+x}\frac{1}{\zeta^2}\left(1-\frac{\sin\pi\zeta}{\pi\zeta}\right)d\zeta=\operatorname{sgn}x.$$

This leads us to

$$\sum_{n=-\infty}^{\infty}|A_n(x)|/|x|=\left|\sum_{n=-\infty}^{\infty}(-1)^n A_n(x)\right|\Big/|x|+o(1) \qquad \text{as} \quad x\to\pm\infty.$$

By Theorem 1

$$\sum_{n=-\infty}^{\infty}(-1)^n A_n(x)\equiv\frac{\pi}{2}x\sin\pi x+\cos\pi x.$$

Thus we obtain

$$\sum_{n=-\infty}^{\infty} |A_n(x)| \begin{cases} \leqslant c_{16}, & \text{if} \quad -1 \leqslant x \leqslant 1, \\ \leqslant c_{16}\,|x|, & \text{if} \quad |x| > 1 \end{cases}$$

and

$$\sum_{n=-\infty}^{\infty} \left| A_n\left(j+\frac{1}{2}\right)\right| \geqslant c_{17}\,|2j+1| \qquad (j=0,\ \pm 1,\ \pm 2,\ldots).$$

Hence, setting

$$\lambda_\tau := \max_{-1 \leqslant x \leqslant 1} \sum_{n=-\infty}^{\infty} |A_n(\tau x)| \tag{55}$$

we see that for $\tau > \pi$

$$c_{17} \leqslant \lambda_\tau/\tau \leqslant c_{16}.$$

Next denote by ξ_τ a point of the unit interval where the maximum in (55) is attained and consider

$$R_\tau(f; x) := \sum_{\nu=-\infty}^{\infty} f\left(\frac{\nu\pi}{\tau}\right) A_\nu(\tau x).$$

Now we can use the method of Erdös and Turán [5, pp. 52–54] (see also Kiš [11, pp. 273–276]) to construct for every given $\alpha \in (0, 1)$ a bounded function $f\colon \mathbb{R} \to \mathbb{R}$ belonging to Lip α such that

$$\limsup_{\tau \to \infty} |R_\tau(f; \xi_\tau)| = \infty.$$

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